

Cramér type moderate deviations for intermediate trimmed means

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Abstract

In this article we establish Cramér type moderate deviation results for (intermediate) trimmed means $T_n = n^{-1} \sum_{i=k_n+1}^{n-m_n} X_{i:n}$, where $X_{i:n}$ – the order statistics corresponding to the first n observations of a sequence X_1, X_2, \dots of i.i.d random variables with *df* F . We consider two cases of intermediate and heavy trimming. In the former case, when $\max(\alpha_n, \beta_n) \rightarrow 0$ ($\alpha_n = k_n/n$, $\beta_n = m_n/n$) and $\min(k_n, m_n) \rightarrow \infty$ as $n \rightarrow \infty$, we obtain our results under a natural moment condition and a mild condition on the rate at which α_n and β_n tend to zero. In the latter case we do not impose any moment conditions on F ; instead, we require some smoothness of F^{-1} in an open set containing the limit points of the trimming sequences $\alpha_n, 1 - \beta_n$.

Keywords: intermediate trimmed means; slightly trimmed sums; asymptotic normality; moderate deviations; large deviations.

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1 Introduction and main results

Let X_1, X_2, \dots be a sequence of independent identically distributed (i.i.d.) real-valued random variables (r.v.'s) with common distribution function (*df*) F , and for each integer $n \geq 1$ let $X_{1:n} \leq \dots \leq X_{n:n}$ denote the order statistics based on the sample X_1, \dots, X_n . Introduce the left-continuous inverse function F^{-1} defined as $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $0 < u \leq 1$, $F^{-1}(0) = F^{-1}(0^+)$, and let F_n and F_n^{-1} denote the empirical *df* and its inverse respectively.

Consider the trimmed mean given by

$$T_n = \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i:n} = \int_{\alpha_n}^{1-\beta_n} F_n^{-1}(u) du, \quad (1.1)$$

where k_n, m_n are two sequences of integers such that $0 \leq k_n < n - m_n \leq n$, $\alpha_n = k_n/n$, $\beta_n = m_n/n$. It will be assumed throughout this paper that

$$k_n \wedge m_n \rightarrow \infty \quad (1.2)$$

as $n \rightarrow \infty$. Here and in the sequel, $a \wedge b := \min(a, b)$, $a \vee b := \max(a, b)$.

The asymptotic properties of trimmed and intermediate (when $\alpha_n \vee \beta_n \rightarrow 0$ along with (1.2)) trimmed sums and means were investigated by many authors. In particular in Csörgő et al. (1988) a necessary and sufficient condition for the existence of $\{a_n\}$, $\{b_n\}$ such that the distribution of the properly normalized trimmed mean $a_n^{-1}(T_n - b_n)$ tends to the standard normal law was obtained, and (using a different approach than in Csörgő et al. (1988)) Griffin and Pruitt (1989) derived an equivalent *iff* condition for asymptotic normality of T_n . Both in Csörgő et al. (1988) and in Griffin and Pruitt (1989) a classical result by Stigler (1973) for the trimmed mean with fixed trimming percentages was extended to the case that the fraction of trimming data is vanishing when n gets large. The second order asymptotic properties (Berry-Esseen type bounds and one-term Edgeworth expansions) for (intermediate) trimmed means were established in Gribkova and Helmers (2006, 2007, 2014); Gribkova (2013). Various aspects of the bootstrap for this kind of statistics were studied, e.g., in Hall and Padmanabhan (1992); Deheuvels et al. (1993); Gribkova and Helmers (2007, 2011) (see also the references therein).

The trimmed sums represent a subclass of L -statistics. A number of highly sharp results on Cramér type large and moderate deviations for L -statistics with smooth on $(0, 1)$ weight functions – that are not applicable for the trimmed sum because of the discontinuity of its weights – was obtained by Vandemaële and Veraverbeke (1982); Bentkus and Zitikis (1990); Aleskeviciene (1991). For the case of heavy truncated L -statistics, when the weight function is zero outside some interval $[\alpha, \beta] \subset (0, 1)$, a result on Cramér type large deviations was first established by Callaert et al. (1982); more recently, the latter result was strengthened in Gribkova (2016), where a different approach than in Callaert et al. (1982) was proposed and implemented.

The aim of this article is to investigate Cramér type moderate deviations for intermediate trimmed means. To the best of our knowledge, this subject has not been studied at all. The case of heavy trimming will be also considered.

Define the population trimmed mean and variance functions

$$\begin{aligned}\mu(u, 1-v) &= \int_u^{1-v} F^{-1}(s) ds, \\ \sigma^2(u, 1-v) &= \int_u^{1-v} \int_u^{1-v} (s \wedge t - st) dF^{-1}(s) dF^{-1}(t),\end{aligned}\tag{1.3}$$

where $0 \leq u < 1-v \leq 1$. Note that $\sigma^2(0, 1) = \text{Var}(X_1)$ whenever $\mathbf{E}X_1^2$ is finite. Here and in the sequel, we use the convention that $\int_a^b = \int_{[a,b)}$ when integrating with respect to the left continuous integrator F^{-1} .

Define the ν -th quantile of F by $\xi_\nu = F^{-1}(\nu)$, $0 < \nu < 1$, and let $W_i^{(n)}$, $i = 1, \dots, n$, denote the X_i Winsorized outside of $(\xi_{\alpha_n}, \xi_{1-\beta_n}]$. In other words

$$W_i^{(n)} = \begin{cases} \xi_{\alpha_n}, & X_i \leq \xi_{\alpha_n}, \\ X_i, & \xi_{\alpha_n} < X_i \leq \xi_{1-\beta_n}, \\ \xi_{1-\beta_n}, & \xi_{1-\beta_n} < X_i. \end{cases}\tag{1.4}$$

To normalize T_n , we define two sequences

$$\mu_n = \mu(\alpha_n, 1 - \beta_n), \quad \sigma_{W,n}^2 = \text{Var}(W_i^{(n)}).\tag{1.5}$$

Note that $\sigma_{W,n}^2 = \sigma^2(\alpha_n, 1 - \beta_n)$ and that μ_n and $\sigma_{W,n}$ are suitable location and scale parameters for T_n when establishing its asymptotic normality (cf. Csörgő et al. (1988)). We will suppose throughout this article that $\liminf_{n \rightarrow \infty} \sigma_{W,n} > 0$ (i.e. that $\xi_{\alpha_n} \neq \xi_{1-\beta_n}$ for all sufficiently large n).

Let Φ denote the standard normal distribution function. Here is our first result on Cramér type moderate deviations for the intermediate trimmed mean.

Theorem 1.1 *Suppose that $\mathbf{E}|X_1|^p < \infty$ for some $p > c^2 + 2$ ($c > 0$). In addition, assume that*

$$\frac{\log n}{k_n \wedge m_n} \rightarrow 0 \quad (1.6)$$

as $n \rightarrow \infty$, and that

$$\alpha_n \vee \beta_n = O((\log n)^{-\gamma}), \quad (1.7)$$

for some $\gamma > 2p/(p-2)$, as $n \rightarrow \infty$. Then

$$\begin{aligned} \mathbf{P}\left(\frac{\sqrt{n}(T_n - \mu_n)}{\sigma_{W,n}} > x\right) &= [1 - \Phi(x)](1 + o(1)), \\ \mathbf{P}\left(\frac{\sqrt{n}(T_n - \mu_n)}{\sigma_{W,n}} < -x\right) &= \Phi(-x)(1 + o(1)), \end{aligned} \quad (1.8)$$

as $n \rightarrow \infty$, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$ ($A > 0$).

We relegate the proof of Theorem 1.1 and other main results to Section 3.

Let us discuss the conditions of Theorem 1.1. Since our results concern the relative error in CLT for T_n , certainly under our assumptions the *iff* condition of the asymptotic normality of T_n given in Csörgő et al. (1988) is satisfied (see the proofs in Section 3). This condition is as follows: for every $t \in \mathbb{R}$

$$\Psi_{j,n}(t) \rightarrow 0, \quad j = 1, 2, \quad (1.9)$$

as $n \rightarrow \infty$, where $\Psi_{j,n}$, $j = 1, 2$, are the auxiliary functions defined in (Csörgő et al., 1988, page 674) which correspond to the trimming of the k_n smallest and m_n largest observations respectively. Consider the first of these functions (the second one is defined similarly). In our notation it is equal to

$$\Psi_{1,n}(t) = \begin{cases} \frac{\alpha_n^{1/2}}{\sigma_{W(n)}} \left\{ F^{-1}\left(\alpha_n + t\sqrt{\frac{\alpha_n}{n}}\right) - F^{-1}\left(\alpha_n\right) \right\}, & |t| \leq \frac{1}{2}\sqrt{\alpha_n n}, \\ \Psi_{1,n}\left(-\frac{1}{2}\sqrt{\alpha_n n}\right), & -\infty < t < -\frac{1}{2}\sqrt{\alpha_n n}, \\ \Psi_{1,n}\left(\frac{1}{2}\sqrt{\alpha_n n}\right), & \frac{1}{2}\sqrt{\alpha_n n} < t < \infty. \end{cases} \quad (1.10)$$

If $\alpha_n \vee \beta_n \rightarrow 0$ and condition (1.7) holds, then our moment assumption, i.e., $\mathbf{E}|X_1|^p < \infty$ for some $p > c^2 + 2$ ($c > 0$), implies (1.9) (see the proof of Theorem 1.1). In contrast, if $\alpha_n \vee \beta_n$ does not tend to zero, then (as it follows from (1.10)) the convergence in (1.9) can happen only under some additional smoothness condition on F^{-1} (cf. condition (1.18) in Theorem 1.3). Thus, assumption (1.7) in Theorem 1.1 means that $\alpha_n \vee \beta_n$ converges to zero fast enough, otherwise the trimming would be occur close to the central region, where a smoothness condition on F^{-1} is required, even for the asymptotic normality only.

As is known, the intermediate trimmed mean T_n can serve as a consistent and robust estimator for $\mathbf{E}X_1$ (whenever it exists), and the results on large and moderate deviations for T_n can be helpful to construct confidence intervals for the expectation of X_1 . In particular the question of whether is possible to replace μ_n in (1.8) by $\mathbf{E}X_1$ is of some practical interest. Our next result concerns the properties of the first two moments of T_n and the possibility of replacing the normalizing sequences in (1.8).

Theorem 1.2 *Suppose that the conditions of Theorem 1.1 hold true. Then*

$$n^{1/2}(\mathbf{E}T_n - \mu_n) = o((\log n)^{-1}), \quad (1.11)$$

$$\frac{\sigma_{W,n}}{\sigma} = 1 + o((\log n)^{-2}), \quad (1.12)$$

$$\frac{\sqrt{\text{Var}(T_n)}}{\sigma_{W,n}/\sqrt{n}} = 1 + o((\log n)^{-1}), \quad (1.13)$$

as $n \rightarrow \infty$. Moreover, μ_n and $\sigma_{W,n}$ in relations (1.8) can be replaced respectively by $\mathbf{E}T_n$ and σ or $\sqrt{n\text{Var}(T_n)}$, without affecting the result.

Furthermore, if in addition

$$\alpha_n \vee \beta_n = o[(n \log n)^{-\frac{p}{2(p-1)}}], \quad (1.14)$$

then

$$n^{1/2}(\mathbf{E}X_1 - \mu_n) = o((\log n)^{-1/2}), \quad (1.15)$$

and μ_n in (1.8) can be replaced by $\mathbf{E}X_1$, without affecting the result.

We now turn to the statement of our results on moderate deviations for T_n in the case of heavy trimming. Define four numbers

$$\begin{aligned} a_1 &= \liminf_{n \rightarrow \infty} \alpha_n, & a_2 &= \limsup_{n \rightarrow \infty} \alpha_n, \\ b_1 &= \liminf_{n \rightarrow \infty} (1 - \beta_n), & b_2 &= \limsup_{n \rightarrow \infty} (1 - \beta_n). \end{aligned}$$

Now, we will assume that

$$0 < a_1, \quad b_2 < 1 \quad \text{and} \quad a_2 < b_1. \quad (1.16)$$

In this case no moment assumptions are needed for the asymptotic normality of T_n and for related properties, whereas some smoothness of F^{-1} at the points where trimming occurs becomes essential (see, e.g., Csörgő et al. (1988); Gribkova and Helmers (2014), see also the discussion after the statement of Theorem 1.1).

Let us introduce two sequences of the auxiliary functions:

$$\begin{aligned} G_n(t) &= F^{-1}\left(\alpha_n + t\sqrt{\frac{\alpha_n \log n}{n}}\right) - F^{-1}(\alpha_n), \\ H_n(t) &= F^{-1}\left(1 - \beta_n + t\sqrt{\frac{\beta_n \log n}{n}}\right) - F^{-1}(1 - \beta_n), \end{aligned} \quad (1.17)$$

$t \in \mathbb{R}$. Note that, for a fixed t , we have $\alpha_n + t\sqrt{\frac{\alpha_n \log n}{n}} = \alpha_n(1 + t\sqrt{\frac{\log n}{k_n}}) = \alpha_n(1 + o(1))$ as $n \rightarrow \infty$ (by (1.16)) and $1 - \beta_n + t\sqrt{\frac{\beta_n \log n}{n}} = 1 - \beta_n(1 + o(1))$. In particular this implies that the functions introduced in (1.17) are well-defined for all sufficiently large n .

Now we are in a position to state our first result for the heavy trimmed means.

Theorem 1.3 *Suppose that condition (1.16) is satisfied. In addition, assume that there exists $\varepsilon > 0$ such that for each $t \in \mathbb{R}$*

$$G_n(t) = O((\log n)^{-(1+\varepsilon)}), \quad H_n(t) = O((\log n)^{-(1+\varepsilon)}), \quad (1.18)$$

as $n \rightarrow \infty$. Then, for each $c > 0$ and $A > 0$, relations (1.8) hold true uniformly in the range $-A \leq x \leq c\sqrt{\log n}$.

Remark 1.1 We notice that G_n differs from the difference in the first line in (1.10) only by the presence of the logarithm under the root sign, the same remark can be applied to H_n . Thus, one can see that condition (1.18) is somewhat stronger than the *iff* condition of asymptotic normality by Csörgő et al. (1988) (cf. (1.9)), but it enables us to obtain the results on moderate deviations for T_n .

Remark 1.2 It is also worth noting that (1.18) holds true if, for instance, the inversion F^{-1} satisfies a Hölder condition of degree d (for some $d > 0$) in an open set containing all limit points of the sequences α_n and $1 - \beta_n$. However, the Hölder condition would be excessive for our present purposes, as it can provide us with the stronger results on large deviations (i.e. the deviations in the range of the form $-A \leq x \leq o(n^{r_d})$, for some $0 \leq r_d \leq 1/6$) in the case of heavy trimming (cf. Gribkova (2016)).

Finally, we state a result on moderate deviations, parallel to Theorem 1.2, but now for the case of heavy trimming. A very mild moment condition will be required now to ensure the existence of the variance of T_n .

Theorem 1.4 *Suppose that the conditions of Theorem 1.3 hold true. In addition, assume that $\mathbf{E}|X_1|^\gamma < \infty$ for some $\gamma > 0$. Then*

$$n^{1/2}(\mathbf{E}T_n - \mu_n) = O((\log n)^{-(1+\varepsilon)}), \quad (1.19)$$

$$\frac{\sqrt{\text{Var}(T_n)}}{\sigma_{W,n}/\sqrt{n}} = 1 + O((\log n)^{-(1+\varepsilon)}), \quad (1.20)$$

as $n \rightarrow \infty$, where ε is as in (1.18).

Moreover, relations (1.8) remain valid, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$, for each $c > 0$ and $A > 0$, if we replace μ_n and/or $\sigma_{W,n}$ in it by $\mathbf{E}T_n$ and $\sqrt{n\text{Var}(T_n)}$ respectively.

2 On our approach

Define a binomial r.v. $N_\nu = \#\{i : X_i \leq \xi_\nu\}$, $0 < \nu < 1$. Set $A_n = F_n(\xi_{\alpha_n}) = N_{\alpha_n}/n$, $B_n = 1 - F_n(\xi_{1-\beta_n}) = (n - N_{1-\beta_n})/n$. Consider the mean $\overline{W}_n = \frac{1}{n} \sum_{i=1}^n W_i^{(n)}$ of i.i.d. Winsorized r.v.'s defined in (1.4).

The next lemma will be crucial for our proofs as it provides us with an approximation of T_n by sums of i.i.d. r.v.'s

Lemma 2.1 *The following representation is valid*

$$T_n - \mu_n = \overline{W}_n - \mathbf{E}\overline{W}_n + R_n, \quad (2.1)$$

where

$$R_n = \int_{\alpha_n}^{A_n} [F_n^{-1}(u) - F^{-1}(\alpha_n)] du - \int_{1-\beta_n}^{1-B_n} [F_n^{-1}(u) - F^{-1}(1-\beta_n)] du. \quad (2.2)$$

In fact, representation (2.1) follows from the first part of the proof of Lemma 2.1 in (Gribkova and Helmers, 2014, cf. relation (2.13)). For the convenience of the reader we present briefly its proof.

Proof. Let $W_{i:n}^{(n)}$ be the order statistics corresponding to the sample $W_i^{(n)}$, $i = 1, \dots, n$. Since $W_{i:n}^{(n)} = X_{i:n}$ for $N_{\alpha_n} + 1 \leq i \leq N_{1-\beta_n}$, we can write

$$\overline{W}_n = \frac{1}{n} \sum_{i=1}^n W_i^{(n)} = A_n \xi_{\alpha_n} + \frac{1}{n} \sum_{i=N_{\alpha_n}+1}^{N_{1-\beta_n}} X_{i:n} + B_n \xi_{1-\beta_n}.$$

Using the latter formula, as a result of the direct and simple computations, we obtain

$$\begin{aligned} T_n - \mu_n - [\overline{W}_n - \mathbf{E}\overline{W}_n] &= \frac{1}{n} \left[\operatorname{sgn}(N_{\alpha_n} - k_n) \sum_{i=(k_n \wedge N_{\alpha_n})+1}^{N_{\alpha_n} \vee k_n} (X_{i:n} - \xi_{\alpha_n}) \right. \\ &\quad \left. - \operatorname{sgn}(N_{1-\beta_n} - (n - m_n)) \sum_{i=((n-m_n) \wedge N_{1-\beta_n})+1}^{N_{1-\beta_n} \vee (n-m_n)} (X_{i:n} - \xi_{1-\beta_n}) \right], \end{aligned} \quad (2.3)$$

where $\operatorname{sgn}(s) = s/|s|$, $\operatorname{sgn}(0) = 0$. The r.h.s. of (2.3) is equal to R_n . The lemma is proved.

Remark 2.1 It is worth noting that under additional smoothness conditions on F^{-1} (in an open set containing the limit points of the sequences α_n and $1-\beta_n$) representation (2.1) can be extended to a U -statistic type approximation for T_n . We refer to Lemma 2.1 in Gribkova and Helmers (2014) for the details. In order to get the quadratic term of the U -statistic approximation, in the cited paper we apply some special Bahadur–Kiefer representations of von Mises statistic type for intermediate sample quantiles obtained in Gribkova and Helmers (2012). It should be also noted that the idea to approximate the trimmed sums by the sums of i.i.d. Winsorized r.v.'s (as a linear term of the approximation) plus a quadratic U -statistic term based on the Bahadur type representations was first proposed and implemented in Gribkova and Helmers (2006, 2007), where the validity of the one-term Edgeworth expansions for a (Studentized) trimmed mean and its bootstrapped version was proved and simple explicit formulas of these expansions were found. This approach can be extended to the case of trimmed L -statistics (cf. Gribkova (2016)).

The following lemma on bounds for absolute moments of order statistics will be applied in the proof of Theorems 1.1-1.4. This lemma was obtained in (Gribkova, 1995, Theorem 1), so we present here only its statement.

Let k and δ be arbitrary positive numbers. Put $\rho = k/\delta$ and set $\alpha_i = i/(n+1)$.

Lemma 2.2 (Gribkova (1995)) *For all $n \geq 2\rho + 1$ and for all i such that $\rho \leq i \leq n - \rho + 1$ the following inequality holds*

$$\mathbf{E}|X_{i:n}|^k < C(\rho) \{[\alpha_i(1 - \alpha_i)]^{-1} \mathbf{E}|X_1|^\delta\}^\rho, \quad (2.4)$$

where one can put $C(\rho) = 2\sqrt{\rho} \exp(\rho + 7/6)$.

Obviously, estimate (2.4) is non-trivial only when $\mathbf{E}|X_1|^\delta < \infty$. We also emphasize the fact that the case $k > \delta$, the most interesting for us here, is allowed in (2.4).

3 Proofs

In this section we prove Theorems 1.1-1.4 stated in Section 1.

Proof of Theorem 1.1. It suffices to prove the first relation in (1.8) (the second relation follows from the first one if we replace X_i by $-X_i$). Define the df 's

$$F_{T_n}(x) = \mathbf{P}\left(\frac{\sqrt{n}(T_n - \mu_n)}{\sigma_{W,n}} < x\right); \quad F_{W_n}(x) = \mathbf{P}\left(\frac{\sqrt{n}(\overline{W}_n - \mathbf{E}\overline{W}_n)}{\sigma_{W,n}} < x\right). \quad (3.1)$$

Applying the classical Slutsky argument to $T_n - \mu_n = \overline{W}_n - \mathbf{E}\overline{W}_n + R_n$ (cf. (2.1)) gives, for $\delta_n > 0$, that $1 - F_{T_n}(x)$ is bounded from above and below by respectively

$$1 - F_{W_n}(x - \delta_n) + \mathbf{P}(n^{1/2}\sigma_{W,n}^{-1}|R_n| > \delta_n)$$

and

$$1 - F_{W_n}(x + \delta_n) - \mathbf{P}(n^{1/2}\sigma_{W,n}^{-1}|R_n| > \delta_n).$$

Now we choose $\delta_n = [\log(1 + n)]^{-d}$ with $d = \gamma \frac{p-2}{2p} - \frac{1}{2}$, where γ is the constant from condition (1.7). Observe that $d > 1/2$ (cf. (1.7)) and hence $\delta_n \sqrt{\log n} \rightarrow 0$ as $n \rightarrow \infty$.

Thus in order to prove our theorem, it suffices to show that

$$1 - F_{W_n}(x \pm \delta_n) = [1 - \Phi(x)](1 + o(1)) \quad (3.2)$$

and

$$\mathbf{P}(n^{1/2}\sigma_{W,n}^{-1}|R_n| > \delta_n) = [1 - \Phi(x)]o(1), \quad (3.3)$$

as $n \rightarrow \infty$, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$ ($A > 0$). Fix an arbitrary $A > 0$.

Let us prove (3.2). Define $Z_i^{(n)} = \sigma_{W,n}^{-1}(W_i^{(n)} - \mathbf{E}W_i^{(n)})$, $i = 1, \dots, n$, $n = 1, 2, \dots$, and note that these r.v.'s form a triangular series of i.i.d. in each series r.v.'s. Further, $\mathbf{E}Z_i^{(n)} = 0$, $\text{Var}Z_i^{(n)} = 1$, and $\mathbf{E}|Z_i^{(n)}|^p$ is bounded from above uniformly in n (because $W_i^{(n)}$ is the Winsorized X_i , and $\mathbf{E}|X_1|^p < \infty$). An application of a now classical result by Rubin and Sethuraman (1965) for scheme series yields that

$$1 - F_{W_n}(x \pm \delta_n) = [1 - \Phi(x \pm \delta_n)](1 + o(1)) \quad (3.4)$$

uniformly for x in the range $0 \leq x \pm \delta_n \leq c_1\sqrt{\log n}$, for every c_1 such that $p > c_1^2 + 2$, where we choose $c_1 > c$. Furthermore, by CLT for $Z_i^{(n)}$, $i = 1, \dots, n$, $n = 1, 2, \dots$, relation (3.4) holds true uniformly in the range $-A_1 \leq x \pm \delta_n \leq 0$ for each $A_1 > 0$. Set $q = \sup_{n \in \mathbb{N}} |\delta_n| = (\log 2)^{-d}$ and put $A_1 = A + q$. Then we obtain the validity of (3.4)

uniformly in the range $-A \leq x \leq c\sqrt{\log n} + r_n$, where $r_n = (c_1 - c)\sqrt{\log n} - q \rightarrow +\infty$ as $n \rightarrow \infty$. At this point we apply Lemma A1 of Vandemaële and Veraverbeke (1982), where the asymptotic property of Φ we need here is given in a convenient form. Since $\delta_n\sqrt{\log n} = o(1)$, due to that lemma we obtain that $1 - \Phi(x \pm \delta_n) = [1 - \Phi(x)](1 + o(1))$, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. Combining (3.4) and the latter arguments, we find that (3.2) is valid uniformly in the required range.

Let us prove (3.3). We first write

$$\mathbf{P}(n^{1/2}\sigma_{W,n}^{-1}|R_n| > \delta_n) \leq \mathbf{P}(n^{1/2}\sigma_{W,n}^{-1}|R_{n,\alpha}| > \delta_n/2) + \mathbf{P}(n^{1/2}\sigma_{W,n}^{-1}|R_{n,\beta}| > \delta_n/2), \quad (3.5)$$

where $R_{n,\alpha}$, $R_{n,\beta}$ denote the first and second integrals in (2.2) respectively. We will prove (3.3) for the first probability on the r.h.s. of (3.5), the treatment for the second one is similar and therefore omitted. In view of our moment assumption, $\sigma_{W,n} \rightarrow \sigma = [\text{Var}(X_1)]^{1/2}$, so it is sufficient to prove that

$$\mathbf{P}(n^{1/2}|R_{n,\alpha}| > L\delta_n) = [1 - \Phi(x)]o(1), \quad (3.6)$$

as $n \rightarrow \infty$, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$, where L stands for a positive constant not depending on n , which may change its value from line to line. Notice that

$$\frac{1}{1 - \Phi(x)} \leq \frac{1}{1 - \Phi(c\sqrt{\log n})} \sim c\sqrt{\log n} n^{c^2/2}, \quad (3.7)$$

for $x \in [-A, c\sqrt{\log n}]$. Hence (3.6) is implied by

$$\mathbf{P}(n^{1/2}|R_{n,\alpha}| > L\delta_n) = o\left((\log n)^{-1/2} n^{-c^2/2}\right). \quad (3.8)$$

Let us prove (3.8). We have $n^{1/2}|R_{n,\alpha}| = \left|n^{-1/2} \sum_{i=(k_n \wedge N_{\alpha_n})+1}^{N_{\alpha_n} \vee k_n} (X_{i:n} - \xi_{\alpha_n})\right|$, and by the monotonicity in i of the difference $X_{i:n} - \xi_{\alpha_n}$ we obtain

$$n^{1/2}|R_{n,\alpha}| \leq n^{-1/2}|N_{\alpha_n} - k_n| |X_{k_n:n} - \xi_{\alpha_n}|. \quad (3.9)$$

Let U_1, \dots, U_n be a sample of independent $(0, 1)$ -uniform distributed r.v.'s, and let $U_{i:n}$ denote the corresponding order statistics. Set $M_{\alpha_n} = \#\{i : U_i \leq \alpha_n\}$. Since the joint distribution of $X_{i:n}$ and N_{α_n} coincides with the joint distribution of $F^{-1}(U_{i:n})$ and M_{α_n} , $i = 1, \dots, n$, we have

$$\mathbf{P}(n^{1/2}|R_{n,\alpha}| > L\delta_n) \leq \mathbf{P}(n^{-1/2}|M_{\alpha_n} - k_n| |F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n)| > L\delta_n) \leq P_1 + P_2, \quad (3.10)$$

where

$$\begin{aligned} P_1 &= \mathbf{P}(|M_{\alpha_n} - k_n| > c_1 \sqrt{k_n \log n}), \\ P_2 &= \mathbf{P}(n^{-1/2} \sqrt{k_n \log n} |F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n)| > L\delta_n), \end{aligned}$$

and c_1 is as before, i.e., $2 + c_1^1 < p$, $c < c_1$. For P_1 by Bernstein's inequality we obtain

$$P_1 \leq 2\exp(-h_n), \quad (3.11)$$

with

$$h_n = \frac{c_1^2 \log n}{2[1 - \alpha_n + (c_1/3)\sqrt{\log n/k_n} (\alpha_n \vee 1 - \alpha_n)]}.$$

Since $\alpha_n \rightarrow 0$ and $\log n/k_n \rightarrow 0$, $n \rightarrow \infty$, we get that $h_n \sim \frac{c_1^2 \log n}{2}$. Hence $h_n > \frac{c_2^2 \log n}{2}$, for some c_2 such that $c < c_2 < c_1$ and all sufficiently large n . Then, relations (3.7) and (3.11) together yield

$$P_1 \leq 2n^{-c_2^2/2} = [1 - \Phi(x)]o(1), \quad (3.12)$$

uniformly in the range $-A \leq x \leq c\sqrt{\log n}$.

It remains to estimate P_2 on the r.h.s. in (3.10). We have

$$\begin{aligned} P_2 &= \mathbf{P}(\alpha_n^{1/2}|F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n)| > L\delta_n/\sqrt{\log n}) \\ &\leq \mathbf{P}(\alpha_n^{1/2}|F^{-1}(U_{k_n:n})| > L\delta_n/\sqrt{\log n} - \alpha_n^{1/2}|F^{-1}(\alpha_n)|) \\ &= \mathbf{P}(\alpha_n^{1/2}|F^{-1}(U_{k_n:n})| > L\delta_n/\sqrt{\log n}[1 - \sqrt{\log n} \delta_n^{-1} \alpha_n^{1/2}|F^{-1}(\alpha_n)|]). \end{aligned} \quad (3.13)$$

Since $\mathbf{E}|F^{-1}(U_1)|^p < \infty$, we have $\alpha_n|F^{-1}(\alpha_n)|^p \rightarrow 0$, $n \rightarrow \infty$, and $\alpha_n^{1/2}|F^{-1}(\alpha_n)| = o(\alpha_n^{\frac{p-2}{2p}})$. From the other hand, $\alpha_n = O((\log n)^{-\gamma})$ (due to condition (1.7)), and $\sqrt{\log n} \delta_n^{-1} \leq (\log(1+n))^{\gamma \frac{p-2}{2p}}$ (by the choice of δ_n). The latter computations yield that the second term within square brackets on the r.h.s. in (3.13) is $o(1)$, and hence it can be omitted. Set $p_n = \mathbf{E}U_{k_n:n} = \frac{k_n}{n+1}$, define $\mathbb{V}_n(p_n) = \sqrt{n}(U_{k_n:n} - p_n)$, and let c_1 be as before (i.e., $2 + c_1 < p$, $c < c_1$). Then we should evaluate

$$\mathbf{P}(\alpha_n^{1/2}|F^{-1}(U_{k_n:n})| > L\delta_n/\sqrt{\log n}) \leq P_3 + P_4, \quad (3.14)$$

$$\begin{aligned} P_3 &= \mathbf{P}(\{\alpha_n^{1/2}|F^{-1}(U_{k_n:n})| > L\delta_n/\sqrt{\log n}\} \cap \{|\mathbb{V}_n(p_n)| \leq c_1\sqrt{\alpha_n \log n}\}), \\ P_4 &= \mathbf{P}(|\mathbb{V}_n(p_n)| \geq c_1\sqrt{\alpha_n \log n}). \end{aligned}$$

In order to estimate P_4 , we can apply Inequality 1 from (Shorack and Wellner, 1986, page 453). Then we obtain

$$P_4 \leq \exp\left[-c_1^2 \frac{\alpha_n \log n}{2p_n}\right] + \exp\left[-c_1^2 \frac{\alpha_n \log n}{2p_n} \tilde{\psi}(t_n)\right], \quad (3.15)$$

where $\tilde{\psi}$ is the function defined in (Shorack and Wellner, 1986, page 453, formula (2)), $t_n = c_1 \frac{\sqrt{\alpha_n \log n}}{p_n \sqrt{n}} = c_1 \sqrt{\frac{n+1}{n}} \sqrt{\frac{\log n}{k_n}}$. By condition (1.6), $t_n \rightarrow 0$ as $n \rightarrow \infty$, hence $t_n > -1$ for all sufficiently large n , and by Proposition 1 in (Shorack and Wellner, 1986, page 455, relation (12)), we find that $\tilde{\psi}(t_n) \geq \frac{1}{1+2t_n/3}$. This and relation (3.15) together yield

$$P_4 \leq 2 \exp\left[-c_2^2 \frac{\log n}{2}\right] = 2n^{-c_2^2/2}, \quad (3.16)$$

for each c_2 such that $c < c_2 < c_1$ and for all sufficiently large n . Let M_n denote $F^{-1}(p_n - c_1 \frac{\sqrt{\alpha_n \log n}}{\sqrt{n}}) \vee F^{-1}(p_n + c_1 \frac{\sqrt{\alpha_n \log n}}{\sqrt{n}})$, then by monotonicity of F^{-1} , we get

$$P_3 \leq \mathbf{P}(\alpha_n^{1/2} M_n > L\delta_n/\sqrt{\log n}). \quad (3.17)$$

Observe that $p_n \pm c_1 \frac{\sqrt{\alpha_n \log n}}{\sqrt{n}} = \alpha_n(1 \pm O(\sqrt{\frac{\log n}{k_n}})) = \alpha_n(1 + o(1))$, and using our moment assumption similarly as before, we find that $\alpha_n^{1/2} M_n = o((\log n)^{-\gamma \frac{p-2}{2p}})$, whereas $\delta_n/\sqrt{\log n} \geq (\log(1+n))^{-\gamma \frac{p-2}{2p}}$. Hence the quantity on the r.h.s. in (3.17) is equal to

zero for all sufficiently large n . Relations (3.10), (3.12)-(3.14) and (3.16)-(3.17) together imply (3.8), which entails (3.3). The theorem is proved.

Proof of Theorem 1.2. Let us first prove (1.11). By Lemma 2.1, we get $\mathbf{E}T_n - \mu_n = \mathbf{E}R_{n,\alpha} - \mathbf{E}R_{n,\beta}$. Here and later on, we keep the notation introduced in the proof of the previous theorem. Then,

$$n^{1/2}|\mathbf{E}T_n - \mu_n| \leq n^{1/2}\mathbf{E}|R_{n,\alpha}| + n^{1/2}\mathbf{E}|R_{n,\beta}|. \quad (3.18)$$

We will estimate only the first term on the r.h.s. in (3.18) (obviously the handling for the second one is similar). As in the proof of Theorem 1.1 (cf. (3.9)-(3.10)), we find that

$$\begin{aligned} n^{1/2}\mathbf{E}|R_{n,\alpha}| &\leq n^{-1/2}\mathbf{E}(|M_{\alpha_n} - k_n| |F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n)|) \\ &\leq n^{-1/2}[\mathbf{E}(M_{\alpha_n} - k_n)^2 \mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2]^{1/2} \\ &= \alpha_n^{1/2}(1 - \alpha_n)^{1/2}[\mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2]^{1/2} \\ &\leq \alpha_n^{1/2}(1 - \alpha_n)^{1/2}2^{1/2}[(\mathbf{E}(F^{-1}(U_{k_n:n}))^2)^{1/2} + |F^{-1}(\alpha_n)|]. \end{aligned} \quad (3.19)$$

By our moment assumption, we have $|F^{-1}(\alpha_n)| = o(\alpha_n^{-1/p})$. In order to estimate $(\mathbf{E}(F^{-1}(U_{k_n:n}))^2)^{1/2}$ on the r.h.s. in (3.19), we apply Lemma 2.2. Then we obtain

$$(F^{-1}(U_{k_n:n}))^2)^{1/2} \leq (\alpha_n(1 - \alpha_n))^{-1/p}(\mathbf{E}|X_1|^p)^{1/p} = O(\alpha_n^{-1/p}).$$

Hence, the quantity on the r.h.s. in (3.19) is of the order $O(\alpha_n^{\frac{p-2}{2p}}) = O((\log n)^{-\gamma \frac{p-2}{2p}}) = o((\log n)^{-1})$. Thus (1.11) follows.

Next we prove (1.15) (using the additional condition (1.14)). We write

$$\begin{aligned} n^{1/2}|\mathbf{E}X_1 - \mu_n| &= n^{1/2} \left| \int_0^{\alpha_n} F^{-1}(u) du + \int_{1-\beta_n}^1 F^{-1}(u) du \right| \\ &\leq n^{1/2} \left(\int_0^{\alpha_n} |F^{-1}(u)| du \right) + n^{1/2} \left(\int_{1-\beta_n}^1 |F^{-1}(u)| du \right). \end{aligned} \quad (3.20)$$

As before, we estimate only the first term on the r.h.s. in (3.20). Since $\alpha_n \rightarrow 0$, our moment assumption implies that for every $K > 0$ and sufficiently large n

$$\begin{aligned} n^{1/2} \int_0^{\alpha_n} |F^{-1}(u)| du &\leq K n^{1/2} \int_0^{\alpha_n} u^{-1/p} du = K n^{1/2} \frac{p}{1-p} \alpha_n^{\frac{p-1}{p}} \\ &= o\left(n^{1/2}(n \log n)^{-1/2}\right) = o\left((\log n)^{-1/2}\right) \end{aligned} \quad (3.21)$$

(here condition (1.14) was applied). Relations (3.20) and (3.21) yields (1.15).

Let us prove (1.12). Since under our moment assumption $\sigma_{W,n} \rightarrow \sigma$ as $n \rightarrow \infty$, it suffices to show that $\sigma^2 - \sigma_{W,n}^2 = o((\log n)^{-2})$. We have

$$\sigma^2 - \sigma_{W,n}^2 = \int_{[0,1]^2 \setminus [\alpha_n, 1-\beta_n]^2} (u \wedge v - uv) dF^{-1}(u) dF^{-1}(v). \quad (3.22)$$

Due to the fact that $\alpha_n < 1/2 < 1 - \beta_n$ for all sufficiently large n and because of the symmetry in our conditions for α_n and β_n , it is sufficient to estimate the integral over the region $0 \leq u \leq \alpha_n$, $u \leq v \leq 1/2$. We have

$$\begin{aligned} & \int_0^{\alpha_n} \int_u^{1/2} (u \wedge v - uv) dF^{-1}(v) dF^{-1}(u) = \int_0^{\alpha_n} u \int_u^{1/2} (1 - v) dF^{-1}(v) dF^{-1}(u) \\ & \leq \int_0^{\alpha_n} u \int_u^{1/2} dF^{-1}(v) dF^{-1}(u) = \int_0^{\alpha_n} u [F^{-1}(1/2) - F^{-1}(u)] dF^{-1}(u) \\ & \leq |F^{-1}(1/2)| \int_0^{\alpha_n} u dF^{-1}(u) + \int_0^{\alpha_n} u |F^{-1}(u)| dF^{-1}(u). \end{aligned} \quad (3.23)$$

For the first integral on the r.h.s. in (3.23) for sufficiently large n we obtain

$$\int_0^{\alpha_n} u dF^{-1}(u) \leq \alpha_n |F^{-1}(\alpha_n)| + K \int_0^{\alpha_n} u^{-1/p} du \leq K \alpha_n^{1-1/p} \frac{2p}{p-1},$$

where K is as before. For the second integral on the r.h.s. in (3.23) similarly we find

$$\int_0^{\alpha_n} u |F^{-1}(u)| dF^{-1}(u) \leq K \int_0^{\alpha_n} u^{\frac{p-1}{p}} dF^{-1}(u) \leq K^2 \alpha_n^{1-2/p} \frac{2p-3}{p-2}.$$

The latter computations imply that the quantity on the r.h.s. in (3.23) is of the order $O(\alpha_n^{\frac{p-2}{p}})$, and by condition (1.7) it is $o((\log n)^{-2})$.

Finally, we prove (1.13). In fact, we should show that

$$n\text{Var}(T_n) - \sigma_{W,n}^2 = o((\log n)^{-1}). \quad (3.24)$$

By Lemma 2.1,

$$\begin{aligned} n\text{Var}(T_n) &= n\text{Var}(\overline{W}_n + R_n) = \sigma_{W,n}^2 + n\text{Var}(R_n) + 2n \text{cov}(\overline{W}_n, R_n) \\ &\leq \sigma_{W,n}^2 + n\text{Var}(R_n) + 2\sigma_{W,n}(n\text{Var}(R_n))^{1/2}. \end{aligned}$$

Hence,

$$n\text{Var}(T_n) - \sigma_{W,n}^2 \leq D_n + 2\sigma_{W,n}D_n^{1/2}, \quad (3.25)$$

where

$$\begin{aligned} D_n &= n\text{Var}(R_n) = n\text{Var}(R_{n,\alpha} - R_{n,\beta}) \\ &\leq n[\text{Var}(R_{n,\alpha}) + \text{Var}(R_{n,\beta}) + 2(\text{Var}(R_{n,\alpha})\text{Var}(R_{n,\beta}))^{1/2}]. \end{aligned} \quad (3.26)$$

It remains to estimate $\text{Var}(R_{n,\alpha})$. Write

$$\begin{aligned} \text{Var}(R_{n,\alpha}) &\leq \mathbf{E}R_{n,\alpha}^2 \leq n^{-2} \mathbf{E}[(M_{\alpha_n} - k_n)^2 (F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2] \\ &\leq n^{-2} \left[\mathbf{E}(M_{\alpha_n} - k_n)^4 \mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^4 \right]^{1/2}. \end{aligned} \quad (3.27)$$

By well-known formula for the forth moment of the binomial r.v., we have $\mathbf{E}(M_{\alpha_n} - k_n)^4 = n\alpha_n(1 - \alpha_n)[1 + 3(n-2)\alpha_n(1 - \alpha_n)] < 3n^2\alpha_n^2[1/k_n + 1] \sim 3n^2\alpha_n^2$. Further, we find that

$$\mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^4 \leq 8[\mathbf{E}(F^{-1}(U_{k_n:n}))^4 + (F^{-1}(\alpha_n))^4],$$

where by our moment assumption we have $(F^{-1}(\alpha_n))^4 = o(\alpha_n^{-4/p})$ and by Lemma 2.2, $\mathbf{E}(F^{-1}(U_{k_n:n}))^4 \leq C[\alpha_n(1 - \alpha_n)]^{-4/p} \mathbf{E}(|X_1|^p)^{4/p} = O(\alpha_n^{-4/p})$. Combining (3.27) and the latter computations, we obtain that $\text{Var}(R_{n,\alpha}) = O(n^{-1}\alpha_n^{1-2/p})$. Similarly, we find that $\text{Var}(R_{n,\beta}) = O(n^{-1}\beta_n^{1-2/p})$. Hence, by (3.26),

$$D_n = O\left((\alpha_n \vee \beta_n)^{1-2/p}\right) = o\left((\log n)^{-2}\right). \quad (3.28)$$

Relations (3.25) and (3.28) imply (3.24). Thus (1.13) follows.

To complete the proof it remains to argue why one can replace the normalizing sequences $\mu_n, \sigma_{W,n}$ in (1.8) as it was stated in Theorem 1.2. In fact, this is implied by Theorem 1.1, Lemma A.1 of Vandemaële and Veraverbeke (1982) and relations (1.11)-(1.13), (1.15). Indeed, let A_n denote $\mathbf{E}T_n$ or – under additional assumption (1.14) – $\mathbf{E}X_1$, and let B_n denote $\sqrt{n\text{Var}(T_n)}$ or σ . Put $\lambda_n = B_n/\sigma_{W,n}$, $\nu_n = n^{1/2}\sigma_{W,n}^{-1}(A_n - \mu_n)$. By (1.11)-(1.13) and (1.15), in each of these options we have $\nu_n = o((\log n)^{-1/2})$ and $\lambda_n - 1 = o((\log n)^{-1})$. Fix an arbitrary $A > 0$ and c_1 such that $c < c_1 < \sqrt{p-2}$. Take $A_1 > A$. By Theorem 1.1,

$$\begin{aligned} \mathbf{P}(B_n^{-1}n^{1/2}(T_n - A_n) > x) &= \mathbf{P}(\sigma_{W,n}^{-1}n^{1/2}(T_n - \mu_n) > \lambda_n x + \nu_n) \\ &= [1 - \Phi(\lambda_n x + \nu_n)](1 + o(1)), \end{aligned} \quad (3.29)$$

for x such that $-A_1 \leq \lambda_n x + \nu_n \leq c_1\sqrt{\log n}$. Note that for all sufficiently large n we have $\lambda_n^{-1}(A_1 - \nu_n) > A$ and $\lambda_n^{-1}(c_1\sqrt{\log n} - \nu_n) > c\sqrt{\log n}$. Hence (3.29) holds uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. Finally, we note that since $\sqrt{\log n}(|\lambda_n - 1|^{1/2} \vee |\nu_n|) \rightarrow 0$ as $n \rightarrow \infty$, the Lemma A.1 of Vandemaële and Veraverbeke (1982) implies that $[1 - \Phi(\lambda_n x + \nu_n)] = [1 - \Phi(x)](1 + o(1))$, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. This fact and relation (3.29) together yield the result desired. The theorem is proved.

Proof of Theorem 1.3. Similarly as in the proof of Theorem 1.1, we start with the application of the Slutsky argument, where now we set $\delta_n = (\log(1+n))^{-(1/2+\varepsilon_1)}$ with an arbitrary ε_1 such that $0 < \varepsilon_1 < \varepsilon$, where ε is as in (1.18). Then, we notice again that it suffices to prove that (3.2) and (3.3) are valid uniformly in the range $-A \leq x \leq c\sqrt{\log n}$ ($A > 0$) (but now for each $c > 0$). Fix arbitrary $c, A > 0$.

First we prove the validity of (3.2). Since $0 < a_1$ and $b_1 < 1$, the r.v.'s $|W_i^{(n)}|$ are bounded from above uniformly in n . Hence the Cramér condition for r.v.'s $W_i^{(n)}$ is satisfied uniformly in n , i.e. $\mathbf{E}e^{h|W_i^{(n)}|} \leq M < \infty$, for some positive M and all $h > 0$ and $n \in \mathbb{N}$. Then an application of a large deviations result for the sum of i.i.d. r.v.'s \overline{W}_n (cf., e.g., Feller (1943); Petrov (1975)) yields

$$1 - F_{W_n}(x \pm \delta_n) = [1 - \Phi(x \pm \delta_n)](1 + o(1)), \quad (3.30)$$

as $n \rightarrow \infty$, uniformly in the range $-A_1 \leq x \pm \delta_n = o(n^{1/6})$. Put $A_1 = A + \sup_{n \in \mathbb{N}} \delta_n = A + (\log 2)^{-(1/2+\varepsilon_1)}$ and notice that $c\sqrt{\log n} = o(n^{1/6} - \delta_n)$. Hence we obtain that (3.30) is valid uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. An application of Lemma A.1 by Vandemaële and Veraverbeke (1982) yields that $[1 - \Phi(x \pm \delta_n)] = [1 - \Phi(x)](1 + o(1))$ uniformly in the range $-A \leq x \leq c\sqrt{\log n}$.

Let us prove that (3.3) is valid uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. As before, we prove it only for $R_{n\alpha}$, i.e. for the first part of R_n . Since $0 < a_1, b_1 < 1$ and because

$\liminf_{n \rightarrow \infty} \sigma_{W,n} > 0$, the variance $\sigma_{W,n}^2 = \text{Var}(W_i^{(n)})$ is bounded away from zero and infinity. So, taking into account (3.7) (cf. also (3.8)), we see that it suffices to prove that

$$\mathbf{P}(n^{1/2}|R_{n,\alpha}| > L\delta_n) = o\left((\log n)^{-1/2}n^{-c^2/2}\right), \quad (3.31)$$

where L denote a positive constant not depending on n , which may change its value. Similarly as in the proof of Theorem 1.1, we write

$$\begin{aligned} \mathbf{P}(n^{1/2}|R_{n,\alpha}| > L\delta_n) &\leq \mathbf{P}(n^{-1/2}|M_{\alpha_n} - k_n||F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n)| > L\delta_n) \\ &\leq \mathbf{P}(|M_{\alpha_n} - k_n| > c_1\sqrt{\alpha_n n \log n}) + \mathbf{P}(\sqrt{\log(n+1)}|F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n)| > L\delta_n), \end{aligned} \quad (3.32)$$

where $c_1 > c$. For the first probability on the r.h.s. in (3.32) we find (cf. (3.11)-(3.12)) that for $c < c_2 < c_1$ and all sufficiently large n it does not exceed $2n^{-c_2^2/2}$ which is $[1 - \Phi(x)]o(1)$, as $n \rightarrow \infty$, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. It remains to evaluate the second probability on the r.h.s. in (3.32). It is equal to

$$\mathbf{P}(|F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n)| > \frac{L}{(\log(1+n))^{(1+\varepsilon_1)}}) \leq P_1 + P_2, \quad (3.33)$$

where

$$\begin{aligned} P_1 &= \mathbf{P}\left(\left\{|F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n)| > \frac{L}{(\log(1+n))^{(1+\varepsilon_1)}}\right\} \cap \left\{|\mathbb{V}_n(\alpha_n)| \leq c_1\sqrt{\alpha_n \log n}\right\}\right), \\ P_2 &= \mathbf{P}\left(|\mathbb{V}_n(\alpha_n)| \geq c_1\sqrt{\alpha_n \log n}\right), \end{aligned}$$

where c_1 is as before (i.e., $c_1 > c$) and $\mathbb{V}_n(\alpha_n) = \sqrt{n}(U_{k_n:n} - \alpha_n)$. By the monotonicity of F^{-1} , we get

$$P_1 \leq \mathbf{P}(|G_n(c_1)| \vee |G_n(-c_1)| > L(\log(1+n))^{-(1+\varepsilon_1)}), \quad (3.34)$$

and by condition (1.18) and due to the fact that $\varepsilon_1 < \varepsilon$, the probability P_1 is zero for all sufficiently large n . Finally, by the same way as in the proof of Theorem 1.1 (cf. (3.15)-(3.16)), an application of Inequality 1 from Shorack and Wellner (1986) yields that $P_2 \leq 2 \exp\left[-c_2^2 \frac{\log n}{2}\right] = 2n^{-c_2^2/2}$ for all sufficiently large n and some c_2 such that $c < c_2 < c_1$. The latter computations imply (3.31). Hence (3.3) holds true as $n \rightarrow \infty$, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$, for each $c > 0$ and $A > 0$. The theorem is proved.

Proof of Theorem 1.4. First we prove (1.19), starting from relation (3.18) and noting that it suffices to evaluate $n^{1/2}\mathbf{E}|R_{n,\alpha}|$. Similarly as in the proof of Theorem 1.2 (cf. (3.19)), we first write

$$\begin{aligned} n^{1/2}\mathbf{E}|R_{n,\alpha}| &\leq n^{-1/2}[\mathbf{E}(M_{\alpha_n} - k_n)^2 \mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2]^{1/2} \\ &= \alpha_n^{1/2}(1 - \alpha_n)^{1/2}[\mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2]^{1/2} \\ &\leq K[\mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2]^{1/2}, \end{aligned} \quad (3.35)$$

where K is some positive constant not depending on n . Set $\mathcal{E} = \{|U_{k_n:n} - \alpha_n| \leq c\sqrt{\frac{\alpha_n \log n}{n}}\}$ and let $\mathbf{1}_{\mathcal{E}}$ denote the indicator of the event \mathcal{E} . Then we can write

$$\begin{aligned} &[\mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2]^{1/2} \\ &= [\mathbf{E}((F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2 \mathbf{1}_{\mathcal{E}}) + \mathbf{E}((F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^2 \mathbf{1}_{\bar{\mathcal{E}}})]^{1/2} \\ &\leq [(G_n^2(c) \vee G_n^2(-c))]^{1/2} + [\mathbf{E}((F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^4)]^{1/4} [\mathbf{P}(\bar{\mathcal{E}})]^{1/4}. \end{aligned} \quad (3.36)$$

By condition (1.18), the first term on the r.h.s. in (3.36) is of the order $O((\log n)^{-(1+\varepsilon)})$. By our moment assumption and Lemma 2.2, the first factor of the second term on the r.h.s. in (3.36) is $O(1)$, and by the Inequality 1 from Shorack and Wellner (1986), for the second factor we get $\mathbf{P}(\bar{\mathcal{E}}) = o(n^{-c^2/8})$. Hence, the second term on the r.h.s. in (3.36) is of negligible order for our purposes and contributes to the first one. The latter estimates and (3.35)-(3.36) imply (1.19).

Finally, we prove (1.20). Similarly to the proof of Theorem 1.2 (cf. (3.24)), we notice that it suffices to show that

$$n\text{Var}(T_n) - \sigma_{W,n}^2 = O((\log n)^{-(1+\varepsilon)}). \quad (3.37)$$

Then, we repeat relations (3.25)-(3.27) from the proof of Theorem 1.2. Thus, we see that one should evaluate the quantity on the r.h.s. in (3.27), but now under the conditions of Theorem 1.4. Similarly as before, we find that $\mathbf{E}(M_{\alpha_n} - k_n)^4 < 3n^2$. So it remains to estimate $[\mathbf{E}(F^{-1}(U_{k_n:n}) - F^{-1}(\alpha_n))^4]^{1/2}$, for which – by the same way as in (3.36) – we get the bound of the order $O((\log n)^{-2(1+\varepsilon)})$. Hence $D_n = n\text{Var}(R_n) = O((\log n)^{-2(1+\varepsilon)})$ (cf. (3.26)), and since $n\text{Var}(T_n) - \sigma_{W,n}^2 = O(D_n^{1/2})$, relation (3.37) follows.

To complete the proof, it remains to argue the possibility of replacing μ_n and $\sigma_{W,n}$ in relation (1.8) by $\mathbf{E}T_n$ and $\sqrt{n\text{Var}(T_n)}$ respectively. Again we set $A_n = \mathbf{E}T_n$, $B_n = \sqrt{n\text{Var}(T_n)}$, $\lambda_n = B_n/\sigma_{W,n}$, $\nu_n = n^{1/2}\sigma_{W,n}^{-1}(A_n - \mu_n)$. Fix arbitrary $c, A > 0$, $A_1 > A$, $c_1 > c$. Then, using Theorem 1.3 and the argument below relation (3.29), we find that (3.29) holds uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. Furthermore, since $\sqrt{\log n}(|\lambda_n - 1|^{1/2} \vee |\nu_n|) \rightarrow 0$ as $n \rightarrow \infty$ (due to (1.19)-(1.20)), by Lemma A.1 of Vandemaële and Veraverbeke (1982), we obtain that $[1 - \Phi(\lambda_n x + \nu_n)] = [1 - \Phi(x)](1 + o(1))$, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. The theorem is proved.

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